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DEC 79 K S AZOURY, B L MILLER N00014-78-C-0428

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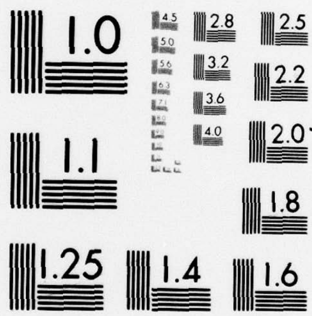

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A COMPARISON OF THE OPTIMAL ORDERING LEVELS OF
BAYESIAN AND NON-BAYESIAN INVENTORY MODELS

by

Katy S. Azoury
College of Business Administration
University of Illinois at Chicago Circle

Bruce L. Miller
Department of System Science
U.C.L.A.

December, 1979

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ABSTRACT

Although it is often the case that the parameters of the distribution of demand are not known with certainty and that a Bayesian formulation would be appropriate, such an approach is generally not used in inventory calculations for computational reasons. Since one often resorts to a non-Bayesian formulation, it is of interest to compare Bayesian policies with a comparable non-Bayesian policy. It was anticipated that the non-Bayesian policy (quantity ordered) would be an upper bound to the Bayesian policy. This result is established for a two-period standard inventory model and for the n-period non-depletive inventory model. However, a counterexample is given in the standard inventory model for an alternative comparable non-Bayesian formulation.

I. INTRODUCTION

The bulk of the development of stochastic dynamic optimization models has been based on the assumption that the underlying probability distribution functions are known with certainty. However, in most real-world applications a certain degree of uncertainty exists about these functions. Being in a dynamic situation provides additional information via observations of the random variables under consideration. Thus, when faced with such uncertainty, it would be desirable if one can incorporate into the optimization model the initial state of knowledge as well as the additional information as it becomes available. A Bayesian approach is often appropriate in these cases, and dynamic programming is the standard solution technique. A Bayesian dynamic program gives rise to adaptive policies which are dependent on the past history. The state space is expanded to include a new state variable that represents past information. In [17], Rieder shows how such an expansion of the state space reduces the Bayesian dynamic program to one with a completely known transition law. The optimal policy may then be determined by solving the resulting optimality equations that are in terms of the augmented state variable.

Among the advantages of a Bayesian approach are the following:

(1) The decision maker can express his/her intuition and previous experience by choosing a prior distribution.

(2) The prior distribution expresses the past experience and prior beliefs in a quantifiable form which can be formally included in the optimization model.

(3) Bayes rule provides a well defined procedure to revise, or update, the prior beliefs as new information becomes available.

However, a Bayesian approach is not without disadvantages. We let $\phi(t|w)$ represent the partially specified distribution with w being the unknown parameter. We also let $g(w)$ denote the prior distribution of w. Bayes rule is

applied to update g(w). In most optimization models the prior distribution g(w) is chosen from the conjugate family for $\phi(t|w)$. This choice is desirable since it results in an updated $g(\cdot)$ within the same family, and hence simplifies its determination. On the other hand, the disadvantage lies in the fact that representation of prior beliefs is restricted to a choice from the conjugate family. This, generally, is not a serious problem when the conjugate family is rich; that is when it includes distributions with different location, dispersion, and shape, so as to represent a wide variety of states of prior beliefs.

A much more serious disadvantage, however, is that the Bayesian dynamic program has a multi-dimensional state variable, and it is rarely cost-effective to compute optimal solutions. In inventory applications this computational intractability is typically "resolved" by assuming that the underlying probability distributions are known with certainty. This non-Bayesian formulation can be thought of as an approximation to the Bayesian inventory model. The question this paper considers is the relationship between the optimal policy of the Bayesian inventory model and the non-Bayesian approximation.

In comparing the Bayesian with the non-Bayesian model a major difference is that further information will be forthcoming in the Bayesian case. Therefore, it appears that one wants to avoid committing oneself in the Bayesian case. In the inventory model committing means purchasing inventory, since if too little is purchased more can be purchased in the next period, while if too much is purchased there is nothing to do but wait for subsequent demand to lower inventory to a proper amount. Therefore, the anticipated result is that the quantity ordered for the Bayesian model is always less than or equal to the non-Bayesian approximation.

The details of the model are presented in Section 2 and two alternatives

for the formulation of the non-Bayesian approximation to the Bayesian model are given. Section 3 is a counterexample which shows the Bayesian model ordering more, using the first alternative formulation of the non-Bayesian approximation. Section 4 establishes the expected result in a two-period model using the second alternative formulation of the non-Bayesian approximation. Section 5 considers the non-depletive inventory model which differs from the standard model in that demand does not reduce the inventory level. Examples are repairable inventory items and capacity expansion models where capacity plays the role of inventory. For the non-depletive model, the expected result is obtained for the n-period model using the first alternatiave formulation of the non-Bayesian approximation.

Other work in Bayesian inventory models has been mainly concerned with deriving the equations of optimality and the characterization of the optimal policy. This work includes Dvoretskey, Kiefer, and Wolfowitz [8], Scarf [19], Karlin [14], Iglehart [13], Hayes [12], Fukuda [11], Van Hee [22], and Waldman [23]. A fascinating computational result was obtained by Scarf [20] who showed how the state variable in the Bayesian inventory model can be reduced to one state variable in the case of linear costs, and a gamma demand with a conjugate gamma prior. This result has been obtained by Azoury [2] for other demand distributions for both the standard and the non-depletive inventory models.

II. The Model

We will consider a finite horizon inventory model with N periods and with a linear ordering, holding, and shortage costs described by parameters c, h, and p respectively. Following Scarf [19] we will usually assume that the demand T is a continuous random variable, and that it may be described by a density $\phi(t|w)$. For expository purposes we will assume that this density is from the exponential class so that

$$\phi(t|w) = b(w)e^{-tw}r(t). \tag{1}$$

We will assume that set of t such that $r(t) > 0$ is convex so that the distribution function of T is strictly increasing. Also we shall assume that an a priori distribution for w, $g(w)$, is given.

At the beginning of the nth period, $n \leq N$, the information available to the decision maker is the present stock level x, and the previous demand observations $t_1,\ldots,t_{n-1}$, which may be summarized in the sufficient statistic $S = \sum_{i=1}^{n-1} t_i/n-1$. The variables x and S will be the state variables of the Bayesian dynamic program.

By Bayes rule the posterior density of w given $(S,n-1)$ is

$$g(w|S,n-1) = \frac{b^{n-1}(w)e^{-w(n-1)S}g(w)}{\int_{-\infty}^{\infty} b^{n-1}(\theta)e^{-\theta(n-1)S}g(\theta)d\theta}. \tag{2}$$

Hence the probability density function of t given $(S,n-1)$ is

$$\phi(t|S,n-1) = r(t)\int_{-\infty}^{\infty} b(\theta)e^{-\theta t}g(\theta|S,n-1)d\theta. \tag{3}$$

If we introduce the notation $\phi_n(t|S) = \phi(t|S,n-1)$, then $\phi_n(t|S)$ represents the probability density of demand that faces the decision maker in period n given that S is the mean of the previous n - 1 demands. Thus from (2) and (3) above we get that

$$\phi_n(t|S) = \frac{r(t)\int_{-\infty}^{\infty} b^n(\theta)e^{-\theta t}e^{-\theta(n-1)S}g(\theta)d\theta}{\int_{-\infty}^{\infty} b^{n-1}(\theta)e^{-\theta(n-1)S}g(\theta)d\theta}. \tag{4}$$

Let $f_n(x,S)$ denote the expected value of discounted costs from period n to the end of the horizon, where the inventory level before ordering is x, the sufficient statistic is S, and an optimal ordering policy is followed. The functional equation for $f_n(x,S)$ is:

$$f_n(x,S) = \min_{y\geq x}\Big\{c(y-x) + L_n(y|S) + \beta\int_0^{\infty} f_{n+1}(y-t,S\circ t)\phi_n(t|S)dt\Big\} \tag{5}$$

with $f_{N+1}(x,S) \equiv 0$, and $L_n(y|S)$, representing the expected one-period holding and shortage costs, given by

$$L_n(y|S) = \begin{cases} \int_0^y h(y-t)\phi_n(t|S)dt + \int_y^\infty p(t-y)\phi_n(t|S)dt & \text{if } y \geq 0 \\ \int_0^\infty p(t-y)\phi_n(t|S)dt & \text{if } y < 0 \end{cases} \tag{6}$$

The parameter β represents the one period discount factor. The notation $S \circ t$ is the updated sufficient statistic which equals $\{(n-1)S+t\}/n = S + (t-S)/n$ for the exponential case. It will cause no difficulty in what follows to allow the parameters c, h and p to vary with the period, or to permit the salvage function, $f_{N+1}(x,S)$, to be linear.

As indicated in the introduction, it is often not cost-effective to solve (5) for the optimal policy and instead a non-Bayesian formulation is solved. This raises the question of what the non-Bayesian demand distribution should be and we will give two possibilities.

For both alternatives of the non-Bayesian models we restrict attention to a planning horizon of two periods, and we assume that the demand random variables are independently distributed. These variables will be denoted by $\bar{T}_1$ and $\bar{T}_2$ with densities $\bar{\phi}_1(\cdot)$ and $\bar{\phi}_2(\cdot)$.

The first non-Bayesian formulation called Case I defines the independent demand distributions to be such that

$$\bar{\phi}_1(t) = \bar{\phi}_2(t) = \int_{-\infty}^{\infty} \phi(t|\omega)g(\omega)d\omega. \tag{7}$$

So, in each period, the demand distribution of this non-Bayesian model equals the Bayes estimate of the unknown demand distribution $\phi(t|\omega)$ with respect to the initial prior. This Case I non-Bayesian model, therefore, assumes that the first estimate of the unknown distribution is the true one. In period 1 the Case I non-Bayesian model and the Bayesian model have identical demand distributions.

The second non-Bayesian model called Case II is formulated as follows.

The distribution of demand $\bar{T}_1$ in period 1 is given by $\bar{\phi}_1(t)$ as defined in (7). In period 2, the demand random variable $\bar{T}_2$ is independent of $\bar{T}_1$ and satisfies

$$\bar{T}_1 + \bar{T}_2 \overset{\mathcal{D}}{=} T_1 + T_2 \tag{8}$$

So, the Bayesian model and the Case II non-Bayesian approximation have equal distributions for both the demand in period 1 and the sum of demands over the two periods. The difference between the two models is that T_1 and T_2 are dependent, while $\bar{T}_1$ and $\bar{T}_2$ are independent. The existence of a random variable $\bar{T}_2$ satisfying (8) is a non-trivial issue. In Section 4, it will be shown that (8) holds for the normal random variable.

For either non-Bayesian case let $\bar{f}_n(x)$ denote the expected value of discounted cost from period n to the end of the horizon where the inventory level before ordering is x, and an optimal ordering policy is followed. The functional equation for $\bar{f}_n(x)$ is:

$$\bar{f}_n(x) = \min_{y \geq x} \left\{ c(y-x) + \bar{L}_n(y) + \beta \int_0^\infty \bar{f}_{n+1}(y-t)\bar{\phi}_n(t)\,dt \right\} \tag{9}$$

with $\bar{f}_{N+1}(x) = 0$, and

$$\bar{L}_n(y) = \begin{cases} \int_0^y h(y-t)\bar{\phi}_n(t)\,dt + \int_y^\infty p(t-y)\bar{\phi}_n(t)\,dt & \text{if } y > 0 \\ \int_0^\infty p(t-y)\bar{\phi}_n(t)\,dt & \text{if } y < 0. \end{cases}$$

III. A Counterexample

This counterexample to the "result" that a Bayesian inventory model will never order more than its non-Bayesian approximation assumes that demand for the non-Bayesian approximation is defined by (7) (Case I). We will allow demand to assume three values $\{0,1,2\}$. Let p_i, i = 0, 1, and 2, represent the probability of demand i, where p_0, p_1, and p_2 are unknown, and $\sum_{i=0}^{2} p_i = 1$. The conjugate prior for p_0, p_1 and p_2 is the Dirichlet distribution which in this

case is described by three parameters N_0, N_1, and N_2. It is known that the Bayes estimate of the probability that demand is i is $N_i/\Sigma_j N_j$. Let i be the observed demand. Then the prior is updated by increasing N_i to $N_i + 1$ and leaving N_j, $j \neq i$, unchanged.

In this counterexample it is crucial that the distributions have very small tails and so we let $N_0 = 1$, $N_1 = 97$, and $N_2 = 1$. For the first period of the Bayesian model and for both periods of the non-Bayesian approximation, the Bayes estimate of the probability that demand equals 0 is 1/99, that demand equals 1, is 97/99, and that demand equals 2 is 1/99.

For the Bayesian model, the following table lists the probability distributions for demand in period 2 for all possible realizations of demand in period 1.

	Demand in period 2: 0	1	2
Given 0	$\frac{2}{100}$	$\frac{97}{100}$	$\frac{1}{100}$
Given 1	$\frac{1}{100}$	$\frac{98}{100}$	$\frac{1}{100}$
Given 2	$\frac{1}{100}$	$\frac{97}{100}$	$\frac{2}{100}$

All costs are linear with the following per unit rates. The ordering cost $c = 3$, the holding cost $h = 0$, the penalty cost $p = 3.009$, and a piecewise linear salvage function $f_3(x) = \min(0,-1.025x)$. It was necessary to choose carefully the probabilities and the cost structure. The difficulty encountered in making the right choice of numbers may be taken as an indication that in most cases the Bayesian critical number in period 1 is not larger than the non-Bayesian one.

Let $J_n(\cdot)$, $n = 1,2$ represent the expected value of immediate costs plus the

minimum expected future costs where x is the starting inventory in period n. As an example say we start with $x = 1$ at the beginning of period 2. At the end of period 2, our inventory level may be either 1 or 0 or -1. For the non-Bayesian model, $J_2(1) = (-1.025)(1/99) + (3.009)(1/99) = 0.020040404$. For the Bayesian model, assuming demand in period 1 was 0, then $J_2(1|0) = (-1.025)(2/100) + (3.009)(1/100) = 0.00959$. The following table gives the values of $J_2(x)$ when $x = 2,1,0,-1$ for both models

Inventory Level

$x =$	2	1	0	-1
Non-Bayesian $J_2(x)$	-1.025	0.020040404	3.009	6.018
Bayesian $J_2(x\|0)$	-1.03525	0.00959	2.97891	5.98791
Bayesian $J_2(x\|1)$	-1.025	0.01984	3.009	6.018
Bayesian $J_2(x\|2)$	-1.01475	0.04993	3.03909	6.04809

The equation of optimality in period n in the non-Bayesian case is

$$\bar{f}_n(x) = \min_{y \geq x} \left\{ c(y-x) + J_n(y) \right\}. \tag{11}$$

The Bayesian equation is

$$f_n(x|i) = \min_{y \geq x} \left\{ c(y-x) + J_n(y|i) \right\}. \tag{12}$$

Using the numbers given in the table above, and recalling that $c = 3$, (11) and (12) show that the optimal ordering level in period 2 is zero for both the non-Bayesian case and the Bayesian cases.

In period 1 there is no demand history and the Bayesian as well as the non-Bayesian model have only one state variable. The values of the J_1 functions are:

	Non-Bayesian	Bayesian
$J_1(2)$	0.039675951	0.03968
$J_1(1)$	3.039505458	3.039703838
$J_1(0)$	9.018	9.018

For example for the Bayesian model

$$J_1(1) = (3.009)(1/99) + (1/99)f_2(1|0) + (97/99)f_2(0|1) + (1/99)f_2(-1|2)$$

$$= 1/99[3.009 + 0.00959 + (97)(3.009) + 6.03909]$$

$$= 3.039703838.$$

The equations of optimality (11) and (12) indicate that in period 1 the Bayesian model with an inventory level of 0 orders 2 while the non-Bayesian model with an inventory level of 0 orders 1.

IV. The Standard Inventory Model

In this section we will consider the two-period model and assume that the demand distribution in period 2 for the non-Bayesian approximation satisfies (8) (Case II). We will show that for the exponential class of distributions satisfying a single-crossing hypothesis, the Bayesian inventory model will never order more than its non-Bayesian approximation. The proof will assume continuous random variables. It appears that the proof can be modified in a straight-forward way to hold for discrete random variables of the exponential class.

The single-crossing property we will need is that there is a function $m(t_1)$, where t_1 is the observed demand in period 1, such that for any given t_1, $-\infty < t_1 < \infty$, the distribution functions of demand in period 2 satisfy the following

$$\begin{aligned} \Phi_2(t|t_1) &\leq \bar{\Phi}_2(t) \qquad t \leq m(t_1) \\ \Phi_2(t|t_1) &> \bar{\Phi}_2(t) \qquad t > m(t_1). \end{aligned} \tag{13}$$

We will show at the end of this section that the normal distribution satisfies (13). In [2] it is also shown that (13) holds for the Poisson distribution.

We note that the concept of a single crossing property has proven quite useful in determining bounds in reliability theory (Barlow and Proschan [3, Theorem 4.2.18]).

We now state two results of Karlin [14] for the Bayesian inventory model.

Fact 1. The optimal return functions $f_n(x,S)$ defined by (5) are continuously differentiable and convex functions of x.

Fact 2. The optimal policy is determined by critical levels $x_n^*(S)$, such that if at the beginning of period n the inventory level is x, and the current sufficient statistic is S, then the optimal policy is to order $\mathrm{Max}(x_n^*(S)-x,0)$.

It is well-known that similar results hold for the non-Bayesian inventory model. The critical levels for the non-Bayesian inventory model depend only on the period n, and will be dentoed by $\bar{x}_n$. The following lemma gives expressions for the derivatives of f_2 and $\bar{f}_2$ with respect to x where f_2 is defined by (5) and $\bar{f}_2$ by (9). Note that we allow the holding and penalty costs to vary with the period.

Lemma 1. $f_2'(x,t_1) = \max\{-c,-p_2+(p_2+h_2)\Phi_2(x|t_1)\}$ and

$$\bar{f}_2'(x) = \max\{-c,-p_2+(p_2+h_2)\Phi_2(x)\}.$$

Proof. From Fact 2 and the equation of optimality (5).

$$f_2'(x,t_1) = \begin{cases} -c & \text{if } x \le x_2^*(t_1) \\ L_2'(x|t_1) & \text{if } x > x_2^*(t_1) \end{cases} \tag{14}$$

where $L_2'(x|t_1) = -\,p_2 + (p_2+h_2)\int_0^x \phi_2(t|t_1)dt$. The critical level $x_2^*(t_1)$ is precisely that value of x such that $-\,c = L_2'(x|t_1)$. Since $L_2'(x|t_1)$ is increasing in x the result is established for $f_2'(x|t_1)$. The same approach also works for $\bar{f}_2'$ which completes the proof.

Lemma 2. In period 1, the optimal critical level for the Bayesian model is less than or equal to that for the non-Bayesian model, $\bar{x}_1 \leq x_1^*$, if for any y,

$$E_{T_1} \max\{-c,-p_2+(p_2+h_2)\Phi_2(y-T_1|T_1)\}$$
$$\geq E_{\bar{T}_1} \max\{-c,-p_2+(p_2+h_2)\bar{\Phi}_2(y-\bar{T}_1)\}.$$

Proof. The critical level x_1^* minimizes $G_1(y) = cy + L_1(y) + \beta E_{T_1} f_2(y-T_1|T_1)$, and $\bar{x}_1$ minimizes $\bar{G}_1(y) = cy + \bar{L}_1(y) + \beta E_{\bar{T}_1} \bar{f}_2(y-\bar{T}_1)$. Since $G_1(y)$ and $\bar{G}_1(y)$ are differentiable and convex in y, then x_1^* is that value of y for which

$$G_1'(y) = c - p_1 + (p_1+h_1)\Phi_1(y) + E_{T_1} f_2'(y-T_1|T_1) = 0$$

and $\bar{x}_1$ is that value of y such that

$$\bar{G}_1'(y) = c - p_1 + (p_1+h_1)\bar{\Phi}_1(y) + E_{\bar{T}_1} \bar{f}_2'(y-\bar{T}_1) = 0.$$

By Lemma 1 and the hypothesis $E_{T_1} f_2'(y-T_1|T_1) \geq E_{\bar{T}_1} \bar{f}_2'(y-\bar{T}_1)$, so that $G_1'(y) \geq \bar{G}_1'(y)$ since $\Phi_1(y) = \bar{\Phi}_1(y)$. Since G_1' and $\bar{G}_1'$ are increasing functions, $x_1^* \leq \bar{x}_1$ which completes the proof.

Therefore, our objective is to show that the hypothesis of Lemma 2 is satisfied. Our strategy will be to show that the random variable $Y = \Phi_2(y-T_1|T_1)$ is more risky in the sense of Rothschild and Stiglitz [18] than the random variable $X = \bar{\Phi}_2(y-\bar{T}_1)$. The related idea of one random variable being more spread than another was introduced by Bessler and Veinott [5] and applied to an inventory problem.

Lemma 3. (Karlin [23]). If $t_1' \geq t_1$, then $\Phi_2(\cdot|t_1') \leq \Phi_2(\cdot|t_1)$.

Lemma 4. The function m in (13) is non-decreasing.

Proof. Let $t_1' > t_1$, we want to show $m(t_1') \geq m(t_1)$. Suppose the contrary, that $m(t_1') < m(t_1)$. Then there exists an x such that $m(t_1') < x \leq m(t_1)$. By (13), $\Phi_2(x|t_1) \leq \bar{\Phi}_2(x)$ and $\Phi_2(x|t_1') > \bar{\Phi}_2(x)$. Hence $\Phi_2(x|t_1') > \Phi_2(x|t_1)$, but by Lemma 3, $\Phi_2(x|t_1') \leq \bar{\Phi}_2(x|t_1)$ for all x when $t_1' > t_1$, a contradiction which completes the proof.

Lemma 5. For every y there is a q(y), possibly infinite, such that

$$\begin{aligned} &\Phi_2(y-t_1|t_1) \le \bar{\Phi}_2(y-t_1) \text{ if } t_1 \ge q(y) \\ &\Phi_2(y-t_1|t_1) > \bar{\Phi}_2(y-t_1) \text{ if } t_1 < q(y). \end{aligned} \tag{15}$$

Comment. Equation (15) differs from (13) in that t_1 is fixed in (13) and varies in (15).

Proof. Let t_1^* be such that $\Phi_2(y-t_1^*|t_1^*) \le \bar{\Phi}_2(y-t_1^*)$. If no such t_1^* exists the lemma holds for $q = +\infty$. By (13), $y - t_1^* \le m(t_1^*)$. By Lemma 4 for any $t_1 > t_1^*$, $m(t_1) \ge m(t_1^*)$ and thus $y - t_1 < y - t_1^* \le m(t_1^*) \le m(t_1)$. We invoke (13) to obtain $\Phi_2(y-t_1 \mid t_1) \le \bar{\Phi}_2(y-t_1)$. Therefore, (15) holds where $q(y) = \inf\{t_1^*:\Phi_2(y-t_1^*) \le \bar{\Phi}_2(y-t_1^*)\}$. The infimum is achieved since $\Phi_2(y-t_1^*|t_1^*)$ and $\bar{\Phi}_2(y-t_1^*)$ are continuous in t_1^*.

Lemma 6. $E_{T_1}\Phi_2(y-T_1|T_1) = E_{\bar{T}_1}\bar{\Phi}_2(y-\bar{T}_1)$.

Proof: $E_{T_1}\Phi_2(y-T_1|T_1) = E_{T_1}P(T_2 \le y-T_1|T_1)$. By the properties of conditional expectation, $E_{T_1}P(T_2 \le y-T_1|T_1) = P(T_1+T_2 \le y)$. Similarly $E_{\bar{T}_1}\bar{\Phi}_2(y-\bar{T}_1) = E_{\bar{T}_1}P(\bar{T}_2 \le y-\bar{T}_1)$.

By the independence of $\bar{T}_1$ and $\bar{T}_2$, $E_{\bar{T}_1}P(\bar{T}_2 \le y-\bar{T}_1) = P(\bar{T}_1+\bar{T}_2 \le y)$. This concludes the proof using (8).

Lemma 7. Let y be a given arbitrary inventory level. For any z,

$$\int_{-\infty}^{z} \Phi_2(y-t_1|t_1)\phi_1(t_1)dt_1 \ge \int_{-\infty}^{z} \bar{\Phi}_2(y-t_1)\bar{\phi}_1(t_1)dt_1 \tag{16}$$

Proof. If $z < q(y)$ then (16) follows from Lemma 5 and the fact that $\phi_1=\bar{\phi}_1$. For $z \ge q(y)$, let $A(z) = \int_{-\infty}^{z}\Phi_2(y-t_1|t_1)\phi_1(t_1)dt_1$ and $B(z) = \int_{-\infty}^{z}\bar{\Phi}_2(y-t_1)\bar{\phi}_1(t_1)dt_1$.

Then
$$A(z) = \int_{-\infty}^{\infty}\Phi_2(y-t_1|t_1)\phi_1(t_1)dt_1 - \int_{z}^{\infty}\Phi_2(y-t_1|t_1)\phi_1(t_1)dt_1$$
$$B(z) = \int_{-\infty}^{\infty}\bar{\Phi}_2(y-t_1)\bar{\phi}_1(t_1)dt_1 - \int_{z}^{\infty}\bar{\Phi}_2(y-t_1)\bar{\phi}_1(t_1)dt_1,$$

and (16) follows from Lemma 6 and Lemma 5.

Lemma 8. Let $X = \bar{\Phi}_2(y-\bar{T}_1)$, have distribution function F and $Y = \Phi_2(y-T_1|T_1)$ have distrubution function G. Then for any α, $0 \leq \alpha \leq 1$

$$\int_{G^{-1}(\alpha)}^{\infty} \omega dG(\omega) \geq \int_{F^{-1}(\alpha)}^{\infty} \omega dF(\omega)$$

where $G^{-1}(\alpha)$ is well-defined since by assumption the distribution function of T_1 is strictly increasing. A similar statement holds for $F^{-1}(\cdot)$.

Proof. We note that Y and X are strictly decreasing in T_1 and $\bar{T}_1$, the former by Lemma 3. Therefore, $Y \geq G^{-1}(\alpha)$ if and only if $T_1 \leq \Phi_1^{-1}(\alpha)$ where Φ_1 is the common distribution function of T_1 and $\bar{T}_1$. In other words, Y will be in the top α percentile exactly when T_1 is in the lower α percentile. Also $X \geq F^{-1}(\alpha)$ if and only if $\bar{T}_1 \leq \Phi_1^{-1}(\alpha)$. Consequently, using Lemma 7,

$$\int_{G^{-1}(\alpha)}^{\infty} \omega dG(\omega) = \int_{-\infty}^{\Phi_1^{-1}(\alpha)} \Phi_2(y-t_1|t_1)\Phi_1(t_1)dt_1 \geq$$

$$\int_{-\infty}^{\Phi_1^{-1}(\alpha)} \bar{\Phi}_2(y-t_1)\bar{\Phi}_1(t_1)dt_1 = \int_{F^{-1}(\alpha)}^{\infty} \omega dF(\omega)$$

which completes the proof.

Definition (Rothschild and Stiglitz [18]).

Let X and Y be two random variables with distribution functions F and G respectively. Then Y is more risky than X if and only if for every z

$$\int_{-\infty}^{z} F(x)dx \leq \int_{-\infty}^{z} G(y)dy. \tag{17}$$

In [18] Rothschild and Stiglitz establish two other equivalent definitions of risk. We now derive a general result for comparing the riskiness of two random variables.

Proposition 1. Let X and Y be two continuous random variables with strictly increasing distribution functions F and G respectively. Also let $E(X) = E(Y)$. Then Y is more risky than X if for every α, $0 \leq \alpha \leq 1$.

$$\int_{G^{-1}(\alpha)}^{\infty} y dG(y) \geq \int_{F^{-1}(\alpha)}^{\infty} x dF(x)$$

Proof. In order to establish (17) for all z it suffices to establish (17) for z such that $\min(F^{-1}(0), G^{-1}(0)) \leq z \leq \max(F^{-1}(1), G^{-1}(1))$. For such z the fact that F and G are continuous and strictly increasing implies that there is an α satisfying either $F^{-1}(\alpha) = z$ or $G^{-1}(\alpha) = z$.

Suppose $F^{-1}(\alpha) = z$ and $G^{-1}(\alpha) \geq F^{-1}(\alpha)$.

$$\int_{G^{-1}(\alpha)}^{\infty} y dG(y) = - [1-G(y)]y \Big]_{G^{-1}(\alpha)}^{\infty} + \int_{G^{-1}(\alpha)}^{\infty} (1-G(y))dy, \text{ and}$$

$$\int_{F^{-1}(\alpha)}^{\infty} x dF(x) = -[(1-F(x)]x \Big]_{F^{-1}(\alpha)}^{\infty} + \int_{F^{-1}(\alpha)}^{\infty} (1-F(x))dx.$$

Applying our hypothesis,

$$(1-\alpha)[G^{-1}(\alpha)-F^{-1}(\alpha)] + \int_{G^{-1}(\alpha)}^{\infty} (1-G(y))\, dy - \int_{F^{-1}(\alpha)}^{\infty} (1-F(x))dx \geq 0.$$

Since $(1-G(y))$ is decreasing,

$$\int_{F^{-1}(\alpha)}^{G^{-1}(\alpha)} (1-G(y))dy \geq [G^{-1}(\alpha) - F^{-1}(\alpha)][1-G(G^{-1}(\alpha))] = [G^{-1}(\alpha) - F^{-1}(\alpha)]\,[1-\alpha].$$

Therefore by combining the last two inequalities,

$$\int_{F^{-1}(\alpha)}^{\infty} (1-G(y))dy \geq \int_{F^{-1}(\alpha)}^{\infty} (1-F(x))dx, \text{ and}$$

$$\int_{-\infty}^{F^{-1}(\alpha)} G(y)dy \geq \int_{-\infty}^{F^{-1}(\alpha)} F(x)dx \quad \text{since } E(x) = E(Y).$$

Now suppose $F^{-1}(\alpha) = z$ and $F^{-1}(\alpha) > G^{-1}(\alpha)$. Since $Ex = Ey$ the hypothesis is equivalent to the inequality $\int_{-\infty}^{F^{-1}(\alpha)} x dF(x) \geq \int_{-\infty}^{G^{-1}(\alpha)} y dG(y)$.

Applying intergration by parts we get

$$xF(x)\Big]_{-\infty}^{F^{-1}(\alpha)} - \int_{-\infty}^{F^{-1}(\alpha)} F(x)dx - yG(y)\Big]_{-\infty}^{G^{-1}(\alpha)} + \int_{-\infty}^{G^{-1}(\alpha)} G(y)dy \geq 0.$$

Since G is increasing, $\int_{G^{-1}(\alpha)}^{F^{-1}(\alpha)} G(y)dy \geq [F^{-1}(\alpha) - G^{-1}(\alpha)]G(G^{-1}(\alpha)) =$ $[F^{-1}(\alpha) - G^{-1}(\alpha)]\alpha$. Combining these two inequalities yields,

$$\int_{-\infty}^{F^{-1}(\alpha)} G(y)dy - \int_{-\infty}^{F^{-1}(\alpha)} F(x)dx \geq 0.$$

The proof where $G^{-1}(\alpha) = z$ is similar and this completes the proof.

Proposition 1 holds where X and Y are integer-valued random variables. In this case when z is not an integer, there is no α such that either $F^{-1}(\alpha)$ or $G^{-1}(\alpha) = z$. However, the above proof works by establishing (17) for $\underline{z}$ and $\bar{z}$ where

$$\underline{z} = \max\{F^{-1}(\alpha):\ F^{-1}(\alpha) < z,\ G^{-1}(\alpha):\ G^{-1}(\alpha) < z\} \text{ and}$$
$$\bar{z} = \min\{F^{-1}(\alpha):\ F^{-1}(\alpha) > z,\ G^{-1}(\alpha):\ G^{-1}(\alpha) > z\}.$$

Then we use the fact that F and G are constant on $(\underline{z},\bar{z})$ to obtain (17) for z. If X and Y have strictly positive probabilities on the integers then $\underline{z}$ and $\bar{z}$ will be the two integers surrounding z.

<u>Theorem 1</u>. In period 1 the optimal critical level for the Bayesian model is less than or equal to that of the non-Bayesian model.

<u>Proof</u>. By Lemma 8 and Proposition 1 the random variable $\Phi_2(y-T_1|T_1)$ is riskier in the sense of Stiglitz and Rothschild [18] than $\bar{\Phi}_2(y-\bar{T}_1)$. In [18] they show that if U is a convex function and Y is riskier than X then $E(U(Y)) \geq E(U(X))$. This is precisely what we need to satisfy the hypothesis in Lemma 2 letting $U(x) = \max(-c,-p_2+(p_2+h_2)x)$.

We conclude this section by giving some results which show that (8) and the single-crossing property are satisfied by the normal distribution. In [2] it is shown that (8) as well as (13) hold for the Poisson distribution. The normal

distribution has the drawback as a demand distribution since there is a strictly positive probability of negative demand. If this probability is sufficiently small, this should not proscribe its use. This difficulty is really no more serious than using an unbounded distribution such as the Poisson in a situation where an absolute bound can be given for demand. In [4] Bather has used a Weiner process to describe demand, and the increment of a Weiner process over any interval is normally distributed.

Example. The Bayesian demand T_n in period n, n = 1,2, is a normal random variable with an unknown mean w and a known variance σ^2. The conjugate prior on w is also normal with mean μ_0 and variance σ_0^2. DeGroot [7] gives the distribution of T_1 and $(T_2|T_1=t_1)$ and Azoury [2] shows that $T_1 + T_2$ is normal with a mean of $2\mu_0$ and a variance of $4\sigma_0^2 + 2\sigma^2$. Therefore, (8) is satisfied with T_1, $T_2|t_1$, $\bar{T}_1$, and $\bar{T}_2$ as given in the table below.

	Period 1	Period 2
Bayesian	$T_1 \sim N(\mu_0, \sigma_0^2+\sigma^2)$	$(T_2\|t_1) \sim N\left(r_1\mu_0+r_2t_1, \sigma^2+\sigma_0^2r_1\right)$
Non-Bayesian	$\bar{T}_1 \sim N(\mu_0, \sigma_0^2+\sigma^2)$	$\bar{T}_2 \sim N(\mu_0, 3\sigma_0^2+\sigma^2)$

where $r_1 = \sigma^2/(\sigma_0^2+\sigma^2)$ and $r_2 = (\sigma_0^2/\sigma_0^2+\sigma^2)$.

The single crossing property of the normal distribution is established by the next two results.

Lemma 9. If F and G are distribution functions of two normal random variables with means μ_F and μ_G, and variances σ_F^2 and σ_G^2 respectively $\left(\text{with } \sigma_F^2 \neq \sigma_G^2\right)$, then there exists an $x_0 \in (-\infty,\infty)$ such that

$$F(x) \leq G(x) \text{ if } x \leq x_0 \text{ and}$$

$$F(x) > G(x) \text{ if } x > x_0$$

if and only if $\sigma_F^2 < \sigma_G^2$.

Proof. Let Z represent the standard normal variable.

$$F(x) = P(Z \leq (x-\mu_F)/\sigma_F) \text{ and } G(x) = P(Z \leq (x-\mu_G)/\sigma_G) .$$

Now $F(x) \leq G(x)$ if and only if $(x-\mu_F)/\sigma_F \leq (x-\mu_G)/\sigma_G$ or $x \leq (\sigma_G\mu_F-\sigma_F\mu_G)/(\sigma_G-\sigma_F)$. Hence $F(x) > G(x)$ if and only if $x > (\sigma_G\mu_F-\sigma_F\mu_G)/(\sigma_G-\sigma_F)$. Therefore

$$x_0 = (\sigma_G\mu_F-\sigma_F\mu_G)/(\sigma_G-\sigma_F). \tag{18}$$

Proposition 2. For any given t_1, there exists an $m(t_1) \in (-\infty,\infty)$ such that $\Phi_2(x|t_1) \leq \bar{\Phi}_2(x)$ when $x \leq m(t_1)$, and $\Phi_2(x|t_1) > \bar{\Phi}_2(x)$ when $x > m(t_1)$.

Proof. This is a direct consequence of Lemma 9 since the variance of $(T_2|t_1)$ is $\sigma^2 + \sigma_0^2\sigma^2/(\sigma_0^2+\sigma^2)$ and variance of $\bar{T}_2$ is $3\sigma_0^2 + \sigma^2$ and clearly $\sigma^2 + \sigma_0^2\sigma^2/(\sigma_0^2+\sigma^2) < \sigma^2 + 3\sigma_0^2$. From (18) we see

$$m(t_1) = \frac{(r_1\mu_0+r_2t_1)\sqrt{\sigma^2+3\sigma_0^2} - \mu_0\sqrt{\sigma^2+\sigma_0^2\sigma^2/(\sigma_0^2+\sigma^2)}}{\sqrt{\sigma^2+3\sigma_0^2}\ \sqrt{\sigma^2+\sigma_0^2\sigma^2/(\sigma_0^2+\sigma^2)}} . \tag{19}$$

Note that $m(t_1)$ in (19) is increasing in t_1 as promised by Lemma 4.

The Non-Depletive Inventory Model

Mathematically, the non-depletive inventory model differs from the standard inventory model in that demands cause shortages but do not decrease the inventory level. A consequence of this hypothesis is that the inventory level is non-decreasing over time. The problem which motivated this analysis is the inventory control of recoverable or repairable items (Sherbrooke [21], Miller [15], Miller and Modarres-Yazdi [16]). Recoverable spare items are purchased to offset a random number of items that will be in repair. At each time point during the period if the random number of items in repair (demand) is greater than the number of spares (inventory level) a shortage results. Next period the spares are available to meet demand and the decision is whether to purchase more spares.

This model is also applicable to capacity expansion with linear costs (Arrow, Beckman, and Karlin (pp. 92-105 in [4]), since demands result in shortages but do not deplete the capacity (inventory).

Rather than being restricted to a comparison between the Bayesian and non-Bayesian models we introduce the computationally valuable idea of a partially Bayesian model of order M or model B_M where $1 \leq M \leq N$. A partially Bayesian model of order M updates the distribution of demand up to period M using demand information of the previous M-1 periods.

Therefore, B_1 is the non-Bayesian model (Case I) and B_N is the Bayesian model. The following exhibits the sequence of demand distribution functions for the models B_M and B_{M+1}.

Periods:	1	2		M	M+1		N
model B_M	Φ_1	Φ_2	...	Φ_M	Φ_M	...	Φ_M
model B_{M+1}	Φ_1	Φ_2	...	Φ_M	Φ_{M+1}	...	Φ_{M+1}

Its computational usefulness comes from the fact that a partially Bayesian model is easier to solve than a Bayesian model (especially when the practical range of the sufficient statistic is expanding), yet be nearly as accurate in cases where the estimate of demand parameters stabilizes quickly, and when most of the ordering takes place in the early periods.

Our objective is to show that when $M_1 < M_2$ the amount ordered with model B_{M_2} is never more than with B_{M_1} for the first M_1 periods. The proof will be given for continuous random variables in the exponential class. In [2] the proof is given for the non-parametric Bayesian case with Dirichlet prior. The proof uses some results of Ferguson [9,10] in non-parametric Bayesian statistics.

Let $f_n^M(x,S)$ denote the expected value of discounted costs from period n to the the end of the horizon for a partially Bayesian model of order M, where the inventory level before ordering is x, the sufficient statistic is S, and an optimal ordering policy is followed.

The optimality equation for the model B_M varies depending on whether the sufficient statistic is updated or not. For periods $n = 1,\ldots, M-1$,

$$f_n^M(x,S) = \min_{y \geq x} \left\{ c_n(y-x) + L_n^M(y|S) + \beta \int_0^\infty f_{n+1}^M(y, S \circ t)\phi_n(t|S)dt. \quad (20)\right.$$

where

$$L_n^M(y|S) = \int_y^\infty p(t-y)\phi_n(t|S)dt, \text{ and } S \circ t = S + (t-S)/n.$$

Since inventory is non-decreasing any holding cost can be included in the the ordering cost by $c_n = c + \sum_{i=n}^{N} \beta^{(i-n)}h$.

For $n = M, \ldots, N$

$$f_n^M(x,S) = \min_{y \geq x} \left\{ c_n(y-x) + L_n^M(y|S) + \beta f_{n+1}^M(y,S) \right\} \quad (21)$$

where

$$L_n^M(y|S) = \int_y^\infty p(t-y)\phi_M(t|S)dt, \quad \text{and} \quad f_{N+1}^M = 0.$$

The main difference between (20) and (5) is that in (20) the starting inventory in period $n + 1$ is y rather than $y - t$. The same applies to (21) when compared to (9).

We begin with a result analogous to Lemma 1.

__Lemma 10__. $\frac{d}{dx} f_n^M(x,S) = \max \left\{ -c, \frac{d}{dx} J_n^M(x,S) \right\}$

where

$$J_n^M(x,S) = L_n^M(x|S) + \beta \int_0^\infty f_{n+1}^M(x, S \circ t)\phi_n(t|S)dt$$

if $n < M$ and for $n \geq M$

$$J_n^M(x,S) = L_n^M(x|S) + \beta f_{n+1}^M(x,S).$$

__Proof__. From Fact 2 given before Lemma 1 and the equations of optimality (20) and (21) there exists $x_n^{*M}(S)$ such that

$$\frac{d}{dx} f_n^M(x|S) = \begin{cases} -c & \text{if } x \leq x_n^{*M}(S) \\ \frac{d}{dx} J_n^M(x,S) & \text{if } x > x_n^{*M}(S) \end{cases}$$

The critical level $x_n^{*M}(S)$ is precisely that value of x such that $-c = \frac{d}{dx} J_n^M(x,S)$. It remains to show that $\frac{d}{dx} J_n^M(x,S)$ is increasing in x. Now for $n \leq M-1$,

$$\frac{d}{dx} J_n^M(x,S) = - p+p\Phi_n(x|S) + \beta \int_0^\infty \frac{d}{dx} f_{n+1}^M(x,S\circ t)\phi_n(t|S)dt. \quad (22)$$

The convexity of f_{n+1}^M (Fact 1) implies that $\frac{d}{dx} f_{n+1}^M(x,S\circ t)$ is increasing in x which completes the proof for $n \leq M-1$ and the proof for $n \geq M$ is the same after modifying $\frac{d}{dx} J_n^M(x,S)$.

Lemma 11. Let T be the demand in period n. For, $1 \leq n \leq M$,

$$E_T\Phi_{n+1}(x|S\circ T)) = \Phi_n(x|S).$$

Proof. This follows as a basic consequence of conditional expectation. Chung [6, Theorem 9.1.5.].

Lemma 12. Let T denote the demand in period M. Then for $k = M + 1,\dots,N$

$$E_T \frac{d}{dx} f_k^{M+1}(x,S\circ T) \geq \frac{d}{dx} f_k^M(x,S).$$

Proof. We begin the induction for $k = N$. By Lemma 10, equation (22), and the fact that $f_{N+1}^{M+1} = 0$,

$$E_T \frac{d}{dx} f_N^{M+1}(x,S\circ T) = E_T \max[-c,-p+p\Phi_{M+1}(x|S\circ T)].$$

Since $U(x) = \max\{-c,-p+px\}$ is a convex function, by Jensen's inequality

$$E_T \frac{d}{dx} f_N^{M+1}(x,S\circ T) \geq \max\{-c,-p+pE_T\Phi_{M+1}(x|S\circ T))\}$$

$$= \max\{-c,-p+p\Phi_M(x|S)\} \text{ by Lemma 11,}$$

$$= \frac{d}{dx} f_N^M(x,S).$$

Therefore, the result holds for $k = N$. Now assume it holds for some k, $M + 1 < k \leq N$, and we show that it holds for $k - 1$. By Lemma 10 and (22)

$$E_T \frac{d}{dx} f_{k-1}^{M+1}(x,S\circ T) = E_T \max\{-c,-p+p\Phi_{M+1}(x|S\circ T) + \beta \frac{d}{dx} f_k^{M+1}(x,S\circ T)\}.$$

By Jensen's inequality

$$E_T \frac{d}{dx} f_{k-1}^{M+1}(x,S\circ T) \geq \max\{-c,p+pE_T\Phi_{M+1}(x|S\circ T) + \beta E_T \frac{d}{dx} f_k^{M+1}(x,S\circ T)\} \geq$$

$$\max\{-c,p+p\Phi_M(x|S) + \beta \frac{d}{dx} f_k^M(x,S)\} = \frac{d}{dx} f_{k-1}^M(x,S).$$

The second inequality uses Lemma 11 and the induction hypothesis Q.E.D.

Lemma 13. For $k = 1,2,\ldots,M$, $\frac{d}{dx} f_k^{M+1}(x,S) \geq \frac{d}{dx} f_k^M(x,S)$.

Proof. For $k = M$ and by Lemma 10,

$$\frac{d}{dx} f_M^{M+1}(x,S) = \max\{-c,-p+p\Phi_M(x|S) + \beta E_T \frac{d}{dx} f_{M+1}^{M+1}(x,S\circ T)\}$$

$$\geq \max\{-c,-p+p\Phi_M(x|S) + \beta \frac{d}{dx} f_{M+1}^M(x,S)\}$$

$$= \frac{d}{dx} f_M^M(x,S),$$ the inequality by Lemma 12.

Now we assume the result holds for some k, where $2 \leq k \leq M$, and show that it holds for k-1.

$$\frac{d}{dx} f_{k-1}^{M+1}(x,S) = \max\{-c,-p+p\Phi_{k-1}(x|S) + \beta E_T \frac{d}{dx} f_k^{M+1}(x,S\circ T)\}$$

and $$\frac{d}{dx} f_{k-1}^M(x,S) = \max\{-c,-p+p\Phi_{k-1}(x|S) + \beta E_T \frac{d}{dx} f_k^M(s,S\circ T)\}$$

By the induction hypothesis we have

$\frac{d}{dx} f_k^{M+1}(x,S\circ t) \geq \frac{d}{dx} f_k^M(x,S\circ t)$ for every t so that

$E_T \frac{d}{dx} f_k^{M+1}(x,S\circ T) \geq E_T \frac{d}{dx} f_k^M(x,S\circ T)$ and hence

$\frac{d}{dx} f_{k-1}^{M+1}(x,S) \geq \frac{d}{dx} f_{k-1}^M(x,S)$, which completes the proof.

The following theorem compares the critical levels for models B_{M+1} and B_M during the first M periods.

Theorem 2. The critical levels satisfy $x_\ell^{*M+1}(S) \leq x_\ell^{*M}(S)$ for $\ell = 1,2,\ldots,M$.

Proof. For $\ell + M$, $x_M^{*M+1}(S)$ satisfies

$$c + \frac{d}{dx} J_M^{M+1}(x,S) = 0 \text{ or}$$

$$c - p + p\Phi_M(x|S) + \beta E_T \frac{d}{dx} f_{M+1}^{M+1}(x,S\circ T) = 0, \tag{23}$$

while $x_M^{*M}(S)$ satisfies $c + \frac{d}{dx} J_M^M(x,S) = 0$ or

$$c - p + p\Phi_M(x|S) + \beta \frac{d}{dx} f_{M+1}^M(x,S) = 0. \tag{24}$$

It follows from Lemma 12 that

$$E_T \frac{d}{dx} f_{M+1}^{M+1}(x,S\circ T) \geq \frac{d}{dx} f_{M+1}^M(x,S)$$

which implies from (23) and (24) and the fact that $\frac{d}{dx} J_M^{M+1}(x,S)$ and $\frac{d}{dx} J_M^M(x,S)$ are non-decreasing functions of x, that $x_M^{*M+1}(S) \leq x_M^{*M}(S)$.

Similarly for $\ell < M$, $x_\ell^{*M+1}(S)$ satisfies (23) substituting ℓ for M since model B_{M+1} updates in period ℓ,

$\ell \leq M$, while $x_\ell^{*M}(S)$ satisfies $c + \frac{d}{dx} J_\ell^M(x,S) = 0$ or

$$c - p + p\Phi_\ell(x|S) + \beta E_T \frac{d}{dx} f_{\ell+1}^M(x,S\circ T) = 0.$$

It follows from Lemma 13 that

$$\frac{d}{dx} f_{\ell+1}^{M+1}(x,S\circ t) \geq \frac{d}{dx} f_{\ell+1}^M(x,S\circ t)$$

for every t, and thus

$$\frac{d}{dx} J_\ell^{M+1}(x,S) \geq \frac{d}{dx} J_\ell^M(x,S)$$

which implies that $x_\ell^{*M+1}(S) \leq x_\ell^{*M}(S)$, and this completes the proof.

For $1 \leq M_1 < M_2 \leq N$, we can now derive a comparison between the critical ordering levels for models B_{M_1} and B_{M_2} during the first M_1 periods.

<u>Corollary 1</u>. For $M_1 > M_2$, $x_\ell^{*M_1}(S) \leq x_\ell^{*M_2}(S)$ for $\ell = 1,2,\ldots,M_1$. Beyond period M_1 they are not comparable.

<u>Proof</u>. Apply Theorem 2 (M_2-M_1) times.

<u>Corollary 2</u>. In period 1, the optimal critical level for the Bayesian model is less than or equal to that of the non-Bayesian model.

<u>Proof</u>. Apply Corollary 1 with $M_1 = 1$ and $M_2 = N$.

The results of Theorem 2 and Corollaries 1 and 2 give a description of the variation in optimal critical levels in terms of the variation in the number of Bayesian updatings of the demand distribution. The following table presents the above results for the various models. The dependence of the critical levels on the current posterior distribution is suppressed, so write x_n^{*M} instead of $x_n^{*M}(S)$.

Model

Period	B_N		B_{N-1}			B_{M+1}		B_M			B_2		B_1
1	x_1^{*N}	$\le$	x_1^{*N-1}	$\le \dots \le$		x_1^{*M+1}	$\le$	x_1^{*M}	$\le \dots$	$\le$	x_1^{*2}	$\le$	x_1^{*1}
2	x_2^{*N}	$\le$	x_2^{*N-1}	$\le \dots \le$		x_2^{*M+1}	$\le$	x_2^{*M+1}	$\le \dots$	$\le$	x_2^{*2}		
.	.					.		.					
.	.					.		.					
.	.					.		.					
M	x_M^{*N}	$\le$	x_M^{*N-1}	$\le \dots \le$		x_M^{*M+1}	$\le$	x_M^{*M}					
M+1	x_{M+1}^{*N}	$\le$	x_{M+1}^{*N-1}	$\le \dots \le$		x_{M+1}^{*M+1}							
.	.												
.	.												
.	.												
N-1	x_{N-1}^{*N}	$\le$	x_{N-1}^{*N-1}										
N	x_N^{*N}												

REFERENCES

1. Arrow, K. J., S. Karlin, and H. Scarf, Studies in the Mathematical Theory of Inventory and Production, Stanford University Press, Stanford, Calif. 1958.

2. Azoury, K. S., "Bayesian Policies for Dynamic Inventory Models," Ph.D. Dissertation, Engineering, U.C.L.A., 1979.

3. Barlow, R. E. and F. Proschan, Statistical Theory of Reliability and Life Testing: Probability Models, Holt, Rinehart, and Winston, Inc., New York, 1975.

4. Bather, J., "A Continuous Time Inventory Model," Tech. Report No. 5, Statistics Department, Stanford, 1965.

5. Bessler, S. and A. F. Veinott, Jr., "Optimal Policy for a Dynamic Multi-Echelon Inventory Model," Naval Research Logistics Quarterly, Vol. 13, (1966), pp. 355-390.

6. Chung, K., A Course in Probability Theory, Harcourt, Brace and Wald, New York, 1968.

7. Degroot, M. H., Optimal Statistical Decisions, McGraw-Hill, Inc., New York, 1970.

8. Dvoretzky, A., J. Kiefer, and J. Wolfowitz, "The Inventory Problem: II. Case of Unknown Distributions of Demand," Econometrica, Vol. 19 (1952), pp. 451-466.

9. Ferguson, T. S., "A Bayesian Analysis of Some Nonparametric Problems," Annals of Statistics, Vol. 1 (1973), pp. 209-230.

10. Ferguson, T. S., "Prior Distributions on Spaces of Probability Measures," Annals of Statistics, Vol. 2 (1974), pp. 615-629.

11. Fukuda, Y., "Estimation Problems in Inventory Control," Ph.D. Dissertation, Mathematics, U.C.L.A., 1960.

12. Hayes, R. H., "Statistical Estimation Problems in Inventory Control," Management Science, Vol. 15 (1969), pp. 686-702.

14. Karlin, S., "Dynamic Inventory Policy with Varying Stochastic Demands," Management Science, Vol. 6 (1960), pp. 231-258.

15. Miller, B. L., " A Multi-Item Inventory Model with a Joint Backorder Criterion," Operations Research, Vol. 19 (1971), pp. 1467-1476.

16. Miller, B. L., and M. Modarres-Yazdi, "The Distribution of Recoverable Inventory Items From a Repair Center When the Number of Consumption Centers is Large," Naval Research Logistics Quarterly, Vol. 25, (1978), pp. 597-604.

17. Rieder, U., "Bayesian Dynamic Programming," Advances in Applied Probability, Vol. 7 (1975), pp. 330-348.

18. Rothschild, M. and J. E. Stiglitz, "Increasing Risk: I. A Definition," J. Economic Theory, Vol. 2 (1970), pp. 225-243.

19. Scarf, H., "Bayes Solutions of the Statistical Inventory Problem," Annals of Mathematical Statistics, Vol. 30 (1959), pp. 490-508.

20. Scarf, H., "Some Remarks on Bayes Solutions to the Inventory Problem," Naval Research Logistics Quarterly, Vol. 7 (1960), pp. 591-596.

21. Sherbrooke, C. C., "METRIC: A Multi-Echelon Technique for Recoverable Item Control," Operations Research, Vol. 16 (1968), pp. 122-141.

22. Van Hee, K. M., "Adaptive Control of Specially Structured Markov Chains," Bonner Mathematische Schriften, No. 98 (1977), pp. 99-116.

23. Waldman, K. H., "On the Numerical Treatment of a Bayesian Inventory Model," Bonner Mathematische Schriften, No. 98 (1977), pp. 117-132.